The Circle Constant τ

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Here I collect some facts about the circle constant τ.

1 Definition of τ

Definition The circle constant $\tau$is defined as the circumference of a circle divided by its radius.

Since the constant πequals the circumference divided by the diameter we have that $\tau = 2\pi$.

2 Continued fraction

Like π the circle constant τ has an expansion as a continued fraction. It is

$$\tau = 6 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{7 + \cfrac{1}{2 + \cfrac{1}{146 + \cfrac{1}{3 + \cfrac{1}{6 + \cdots}}}}}}}}$$

This leads to the following rational approximations of τ.

Approximation	described by	year	Dec. exp.
6	The Bible	1st millenium BC	6
$6\,\frac{1}{3}$			6.33
$6\,\frac{1}{4}$	Babylonian math		6.25
$6\,\frac{2}{7}$	Archimedes	3rd century BC	6.28571
$6\,\frac{15}{53}$			6.28302
$6\,\frac{32}{113}$	Zǔ Chōngzhī	5th century AD	6.283185841
$6\,\frac{4687}{16551}$			6.2831853060
$6\,\frac{14093}{49766}$			6.283185307237
$6\,\frac{89245}{315147}$			6.2831853071741

3 Decimal expansion

The first part of the digital expansion of τ is

$$\begin{aligned}\tau = 6.&2831853071\,7958647692\,5286766559\,0057683943\,3879875021\\ &1641949889\,1846156328\,1257241799\,7256069650\,6842341359\\ &6429617302\,6564613294\,1876892191\,0116446345\,0718816256\end{aligned}$$

4 Angle measurements

When angles are measured in radians τ corresponds to one turn or (360°). Similarly $\tau/2$corresponds to a halfturn (straight angle, 180°), and $\tau/4$ corresponds to a quaterturn (right angle, 90°). Thus using τ there is no dichotomy between measuring angles in radians and measuring angles in turns. A turn can be subdivided in centiturns (cτ) and milliturns (mτ). A centiturn corresponds to 3.6°, which can also be written as 3°36′. A milliturn corresponding to an angle of 0.36°, which can also be written as 21’36".

Pie charts illustrate proportions of a whole as fractions of a turn. Each one percent is shown as an angle of one centiturn.

Angles can be measured in centiturns or milliturns by use of a centiturn or milliturn protractor.

5 Formulas

A lot of formulas simplify by using τ instead of π. Below you will find some important formulas. I have used a happy smiley if a formula has simplified and a sad smiley if a formula has become more complicated. No smiley means that there have not been any significant change in complexity.

5.1 Geometry

Circumference of a circle :-)

$$c = \tau \cdot r.$$

Area of disc (become more intuitive)

$$\frac{1}{2}\tau \cdot r^2.$$

Surface area of sphere

$$2\tau \cdot r^2.$$

Volume of ball

$$\frac{2}{3}\tau \cdot r^3.$$

Surface of a spherical segment :-)

$$\tau h r.$$

Volume of spherical sector :-)

$$\frac{\tau h r^2}{3}$$

Volume of spherical segment :-(

$$\frac{\tau}{6}h^2\left(3r-h\right)$$

Surface area of torus :-)

$$\tau^2 r_1 r_2.$$

Volume of torus

$$\frac{\tau^2}{2} r_1 r_2^2.$$

Volume of hyper sphere in d dimensions :-(

$$\frac{\left(\frac{\tau}{2}\right)^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}.$$

For d even this reduces to

$$\frac{\tau^{d/2}}{d\left(d-2\right)\left(d-4\right)\ldots 2}.$$

For d odd this reduces to :-)

$$\frac{2\tau^{(d-1)/2}}{d!!}.$$

5.2 Probability and statistics

Density of Gaussian distribution :-)

$$\frac{1}{\tau^{1/2}}\exp\left(-\frac{x^2}{2}\right).$$

Asymptotic minimax redundancy of d dimensional variable :-)

$$\frac{d}{2}\ln\left(n\right)+\frac{1}{2}\ln\left(\tau \mathrm{e}\right).$$

5.3 Trigonometric function and harmonic analysis

Periodicity of trigonometric functions :-)

$$\sin(x+\tau) = \sin(\tau)\,,$$
$$\cos(x+\tau) = \cos(x)\,,$$
$$\tan\left(x+\tfrac{\tau}{2}\right) = \tan(x)\,.$$

Fourier series :-)

$$f(x) = \textstyle\sum_{n=-\infty}^{\infty} c_n \cdot e^{inx},$$
$$c_n = \tfrac{1}{\tau}\textstyle\int_0^{\tau} f(x)\, e^{-inx}\, \mathrm{d}x.$$

Fourier integrals :-)

$$f(t) = \tfrac{1}{\tau}\textstyle\int_{-\infty}^{\infty} F(\omega) e^{i\omega t}\, \mathrm{d}\omega,$$
$$F(\omega) = \textstyle\int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, \mathrm{d}t.$$

5.4 Complex analysis

Cauchy's integral formula :-)

$$f(z_0) = \frac{1}{\tau i}\oint \frac{f(z)}{z-z_0}\,\mathrm{d}z.$$

Stirling's formula :-)

$$n! \approx (\tau e)^{1/2}\, n^n e^{-n}.$$

5.5 Physics

Planck's constant :-)

$$\hbar = \frac{h}{\tau}.$$

Harmonic oscillator :-)

$$\omega = \frac{T}{\tau}.$$

Relativity

$$R_{ik} - \frac{g_{ik} R}{2} + \Lambda g_{ik} = \frac{4\tau G}{c^4} T_{ik}.$$

The cosmological constant Λ from Einstein's field equation is related to the intrinsic energy density of the vacuum ρ_{vac} via the gravitational constant G as follows

$$\Lambda = 4\tau G\ \rho_{\mathrm{vac}}.$$

Coulomb's law for the electric force, describing the force between two electric charges, q_1 and q_2, separated by the distance r (with ϵ_0 representing the vacuum permittivity of free space)

$$F = \frac{|q_1 q_2|}{2\tau\varepsilon_0 r^2}.$$

Magnetic permeability of free space relates the production of a magnetic field in a vacuum by an electric current in units of Newtons (N) and Amperes (A):

$$\mu_0 = 2\tau \cdot 10^{-7}\,\mathrm{N/A}^2.$$

Kepler's third law constant, relating the orbital period (P) and the semimajor axis (a) to the masses (M and m) of two co-orbiting bodies :-)

$$\left(\frac{\tau}{P}\right)^2 a^3 = \omega^2 a^3 = G(M+m).$$

6 Other meanings of τ

Like all other letters in the Latin and Greek alphabet the letter τ is used in different ways in differnt parts of mathematics and physics.

- Ramanujan's tau function in number theory.
- The golden ratio 1.618... (although $\phi$is more common).
- Used as the symbol for tangent by Richard Feynman in order to avoid any confusion caused by "tan"
- Kendall's tau rank correlation coefficient as a non-parametric correlation measure in statistics.
- Sequence A002183 in the OEIS: Number of divisors of highly composite numbers
- Torque, the rotational force in mechanics. Also called moment and denoted M. Torque and moment are vectors so τ or M are normally in bold face or are equipped with a vector arrow.
- Shear stress in continuum mechanics. This is a tensor denoted $\tau_{xy}$when it is stress along the xy plane.
- The symbol for tortuosity in hydrogeology. There are several competing definitions of the concept of tortuosity.
- The symbol of continuity in electro-flux resonance.
- The elementary tau lepton particle in particle physics. The tau leptons are denoted τ^- and τ^+.
- In the physical sciences, tau is sometimes used as time variable, to avoid confusing t as temperature. Examples are: The lifetime of a spontaneous emission process; the time constant of any device, such as an RC circuit; proper time in relativity.
- Tau in astronomy is a measure of opacity, or how much sunlight cannot penetrate the atmosphere.

- The prefix of many stars, via the Bayer stellar designation system. (Tau Ceti is such a star.)
- The perceived motion-gap in General Tau Theory, a psychological principle of perception.
- The expressed period of the freerunning rhythm of an animal (circadian rhythm terminology), i.e., the length of the daily cycle of an animal when kept in constant light or constant darkness.
- Tuning and Analysis Utilities (TAU) performance evaluation tool, computer science.
- The specific tax amount.
- The dose interval in pharmacokinetics.
- Tau in biochemistry is a protein associated with microtubules and is implicated in certain neurodegenerative diseases.

7 History

In 1697 David Gregory used $^{\pi}/_{\rho}$ to denote the circumference of a circle divided by its radius, though $^{\pi}/_{\delta}$ had been used by Oughtred in 1647 for the ratio of circumference to diameter. The first use of π on its own with its present meaning was by William Jones in 1706, and Leonard Euler adopted the symbol in 1737. Because of Euler's prestige mathematicians have followed Euler in the use of π.

Some mathematicians have used 2π as if it was one symbol. For instance H. Laurant wrote $2\pi/4$ rather than $\pi/2$[Lau89].

The idea of using centiturns and milliturns as units was introduced by the British astronomer and science writer Sir Fred Hoyle.

Robert Palais wrote an article in 2001 entitled "Pi is wrong!" [Pal01] where he proposed to have a symbol and suggested the symbol $\pi\!\!\pi$. This symbol did not get any popularity because it is only possible to write it with LaTeX. In 2010 Michael Hartl launced a Tau Manifesto where he advocated for using τ instead [Har10].

The circle constant τ may have been recognized much earlier. The earliest evidenced conscious use of an accurate approximation for the length of a circumference with respect to its radius is in the designs of the Old Kingdom pyramids in Egypt. The Great Pyramid at Giza, constructed c. 2550–2500 BC, was built with a perimeter of 1760 cubits and a height of 280 cubits; the ratio $^{1760}/_{280} = 6.2857 \approx \tau$. Egyptologists such as Professors Flinders Petrie and I. E. S. Edwards have shown that these circular proportions were deliberately chosen for symbolic reasons by the Old Kingdom scribes and architects. The same proportions were used earlier at the Pyramid of Meidum c.2600 BC and later from the the pyramide of Niuserre at Abusir. This application is archaeologically evidenced, whereas no mathematical text on this has survived from the Old Kingdom.

The word *turn* originates from Old English *tyrnan* and *turnian.* It comes from Medieval Latin *tornare*, from Latin, to turn on a lathe, from Greek $\tau o \rho \nu o \sigma$ a lathe. The word was influenced by Anglo-French *turner, tourner* to turn, from Medieval Latin *tornare*, akin to Latin *tenere* to rub.

The geometric notion of a turn has its origin in the sailors terminology of knots where a turn means one round of rope on a pin or cleat, or one round of a coil. For knots the English terms of single turn, round turn and double round turn do not translate directly into the geometric notion of turn, but in German the correspondence is exact.

References

[Har10] M. Hartl. The tau manifesto. Website, June 28 2010.

[Lau89] H. Laurant. *Traité D'Algebra.* 4th. edition edition, 1889.

[Pal01] R. Palais. π is wrong. *Mathematical Intelligenser*, 23(3):78, 2001.